
Vectors and Scalars

AP Physics C

Scalar

A **SCALAR** is ANY quantity in physics that has **MAGNITUDE**, but NOT a direction associated with it.

Magnitude – A numerical value with units.

Scalar Example	Magnitude
Speed	20 m/s
Distance	10 m
Age	15 years
Heat	1000 calories

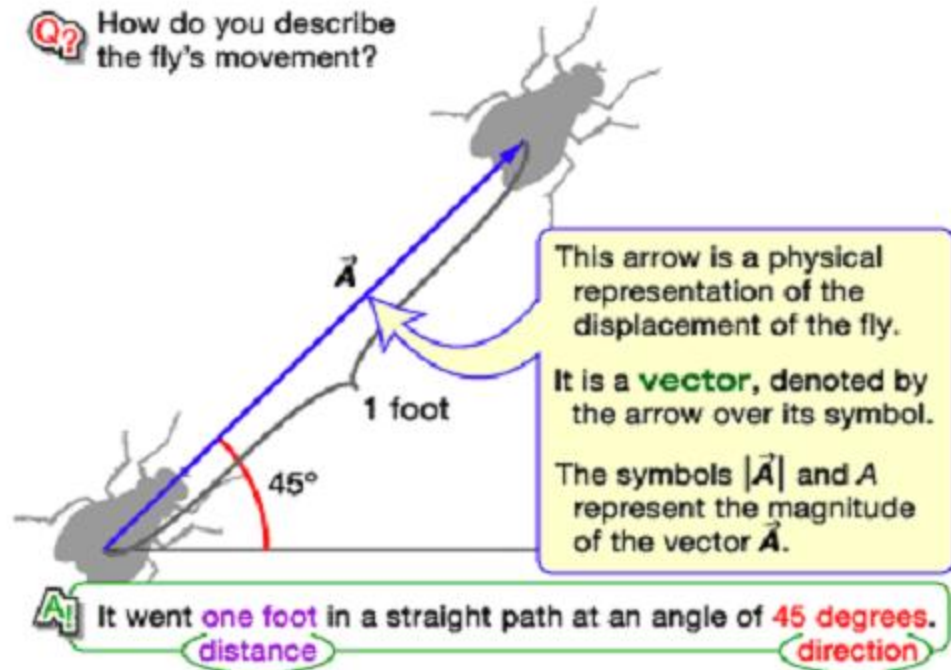
Vector

A **VECTOR** is ANY quantity in physics that has BOTH **MAGNITUDE** and **DIRECTION**.

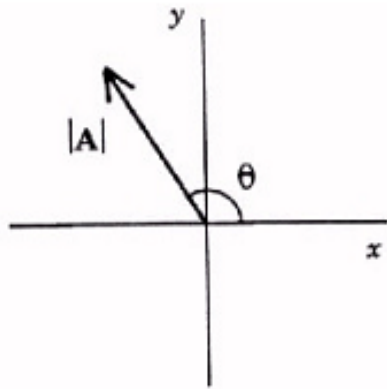
$$\vec{v}, \vec{x}, \vec{a}, \vec{F}$$

Vectors are typically illustrated by drawing an **ARROW** above the symbol. The arrow is used to convey direction and magnitude.

Vector	Magnitude & Direction
Velocity	20 m/s, N
Acceleration	10 m/s/s, E
Force	5 N, West

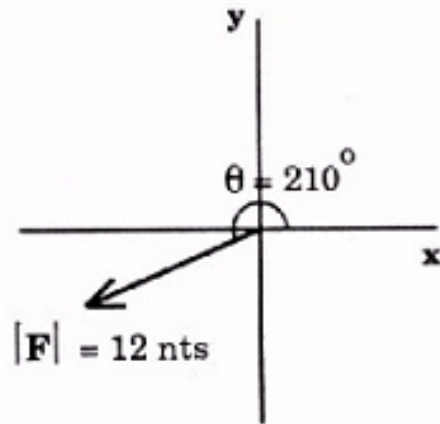


Polar Notation



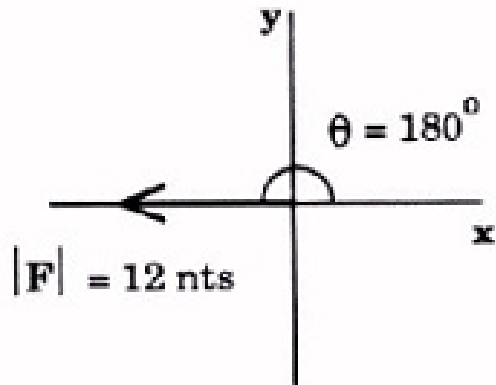
Polar notation defines a vector by designating the vector's magnitude $|A|$ and angle θ relative to the +x axis. Using that notation the vector is written:

$$A = |A| \angle \theta$$



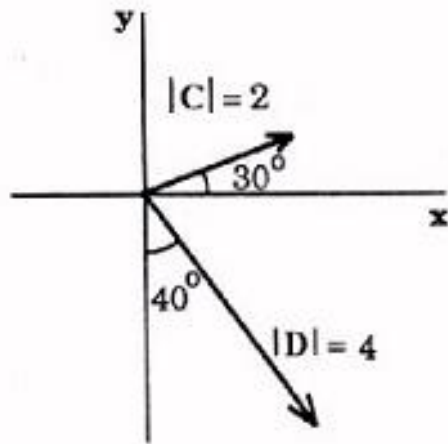
In this picture we have a force vector with magnitude 12 Newtons oriented at 210 degrees with the + x axis. It would be characterized as **$F = 12 < 210$**

Polar Notation



In this picture we have a force vector of 12 Newtons oriented along the -x axis. However, polar notation is relative to the + x axis. Therefore, it would be characterized by

$$F = 12 < 180$$



In this last picture we have 2 vectors. They are characterized by:

$$C = 2 < 30$$

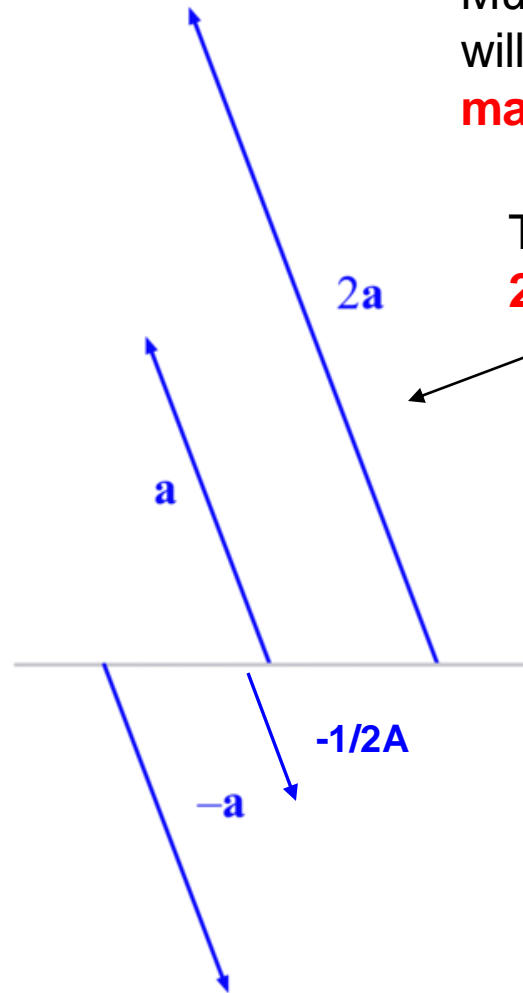
$$D = 4 < -50 \text{ or } D = 4 < 310$$

Scalar Multiplication

Multiplying a vector by a scalar will ONLY CHANGE its **magnitude**.

Multiplying a vector by “-1” does not change the magnitude, but it does reverse it's direction or in a sense, it's angle.

Thus if $A = 12 < 105$, Then $2A = 24 < 105$

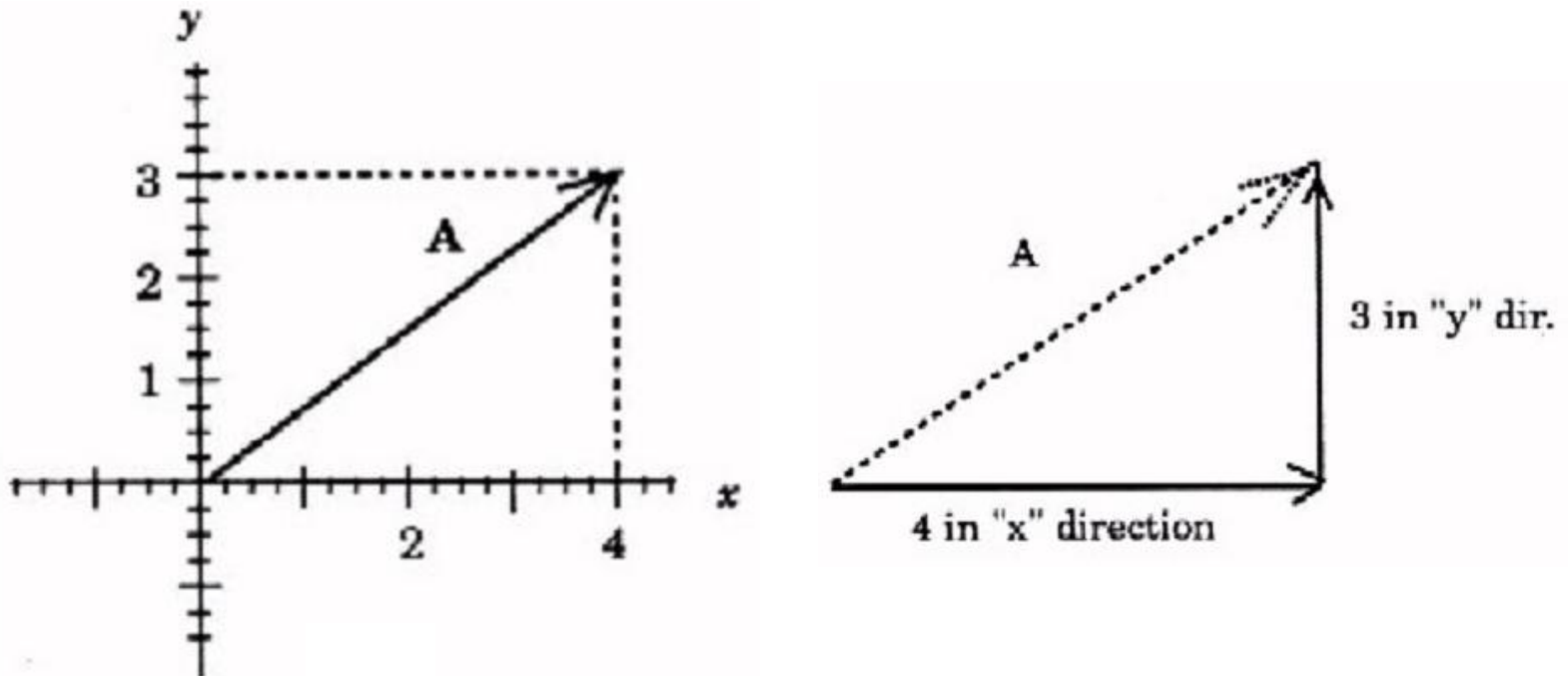


Thus if $A = 12 < 105$, then $-A = 12 < 285$

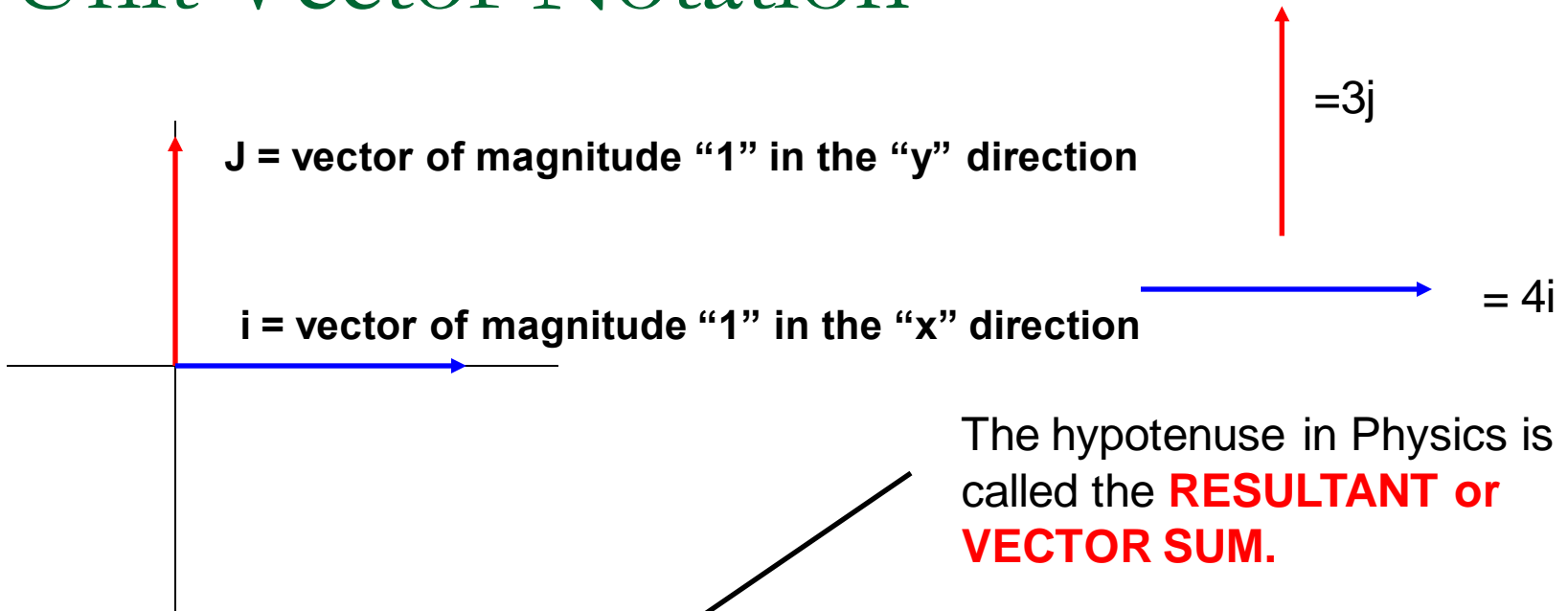
If $A = 12 < 105$, then $(-1/2)A = 6 < 285$

Unit Vector Notation

An effective and popular system used in engineering is called **unit vector notation**. It is used to denote vectors with an x-y Cartesian coordinate system.

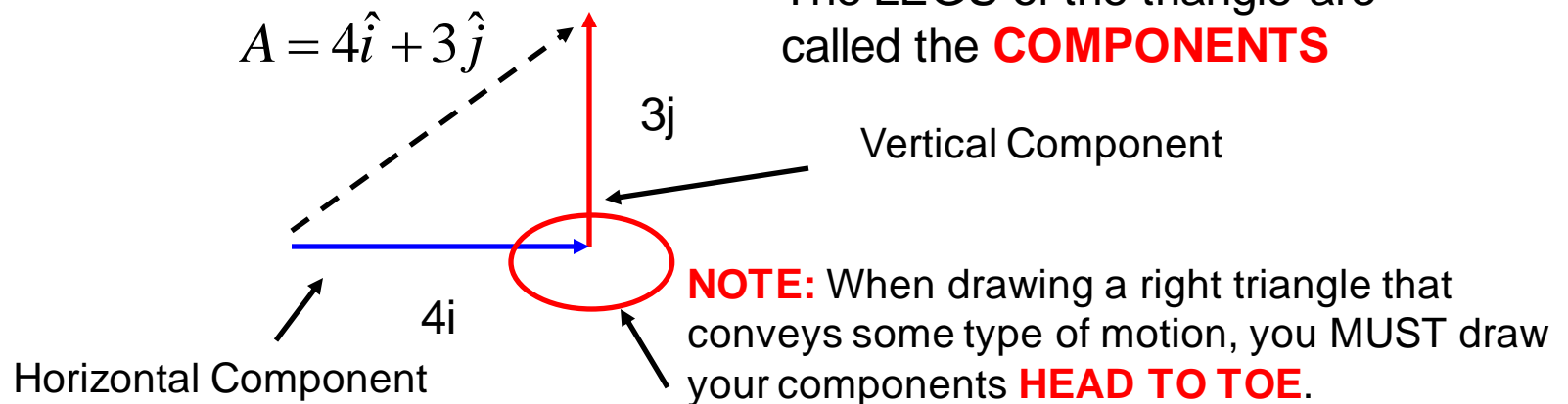


Unit Vector Notation



The hypotenuse in Physics is called the **RESULTANT** or **VECTOR SUM**.

The LEGS of the triangle are called the **COMPONENTS**



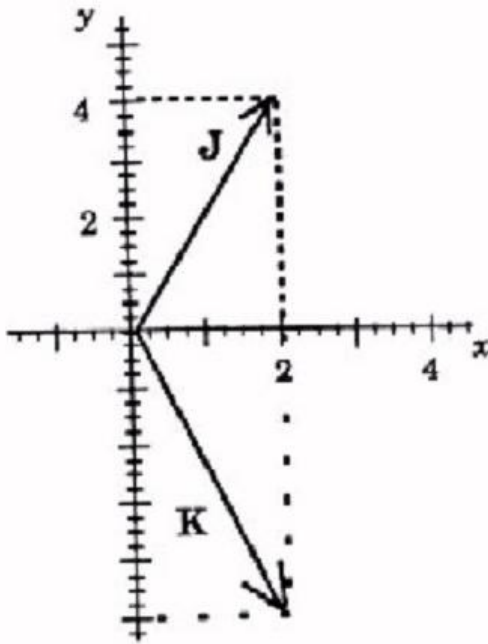
Unit Vector Notation

\hat{i} - unit vector = 1 in the + x direction

\hat{j} - unit vector = 1 in the + y direction

\hat{k} - unit vector = 1 in the + z direction

The proper terminology is to use the “hat” instead of the arrow. So we have **i-hat**, **j-hat**, and **k-hat** which are used to describe any type of motion in 3D space.



How would you write vectors J and K in unit vector notation?

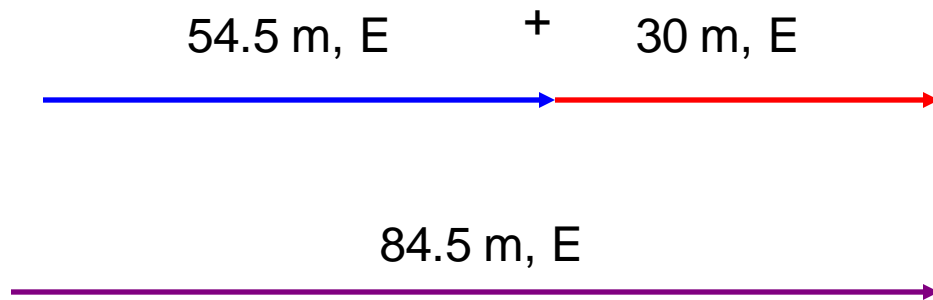
$$J = 2\hat{i} + 4\hat{j}$$

$$K = 2\hat{i} - 5\hat{j}$$

Applications of Vectors

VECTOR ADDITION – If 2 similar vectors point in the SAME direction, add them.

- **Example: A man walks 54.5 meters east, then another 30 meters east. Calculate his displacement relative to where he started?**

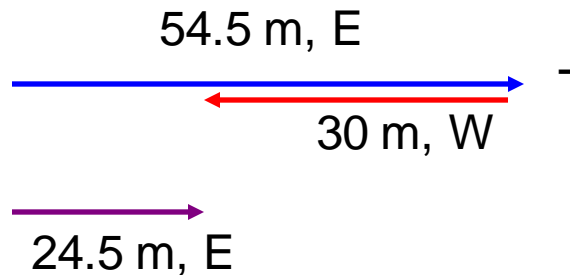


Notice that the SIZE of the arrow conveys **MAGNITUDE** and the way it was drawn conveys **DIRECTION**.

Applications of Vectors

VECTOR SUBTRACTION - If 2 vectors are going in opposite directions, you **SUBTRACT**.

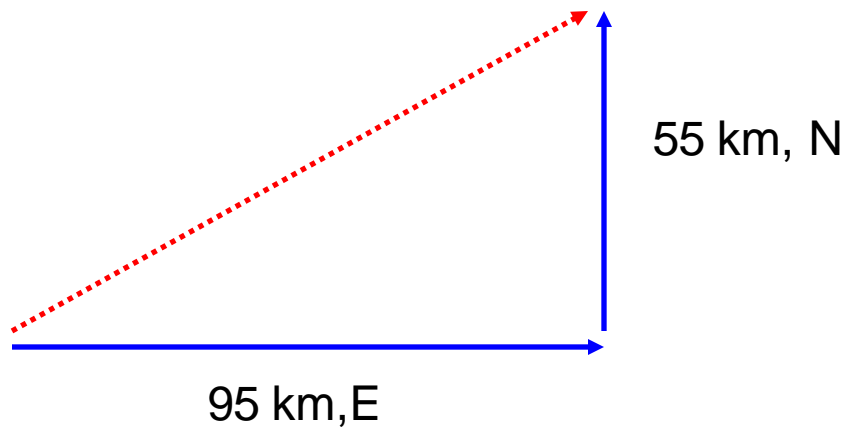
- **Example: A man walks 54.5 meters east, then 30 meters west. Calculate his displacement relative to where he started?**



Non-Collinear Vectors

When 2 vectors are **perpendicular**, you must use the **Pythagorean theorem**.

A man walks 95 km, East then 55 km, north. Calculate his **RESULTANT DISPLACEMENT**.



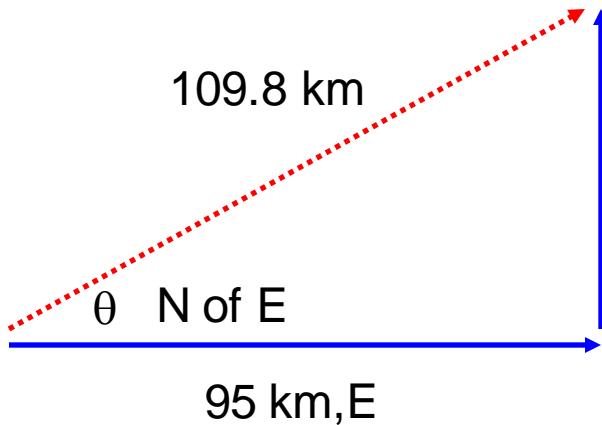
$$c^2 = a^2 + b^2 \rightarrow c = \sqrt{a^2 + b^2}$$

$$c = \text{Resultant} = \sqrt{95^2 + 55^2}$$

$$c = \sqrt{12050} = 109.8 \text{ km}$$

BUT.....what about the VALUE of the angle???

Just putting North of East on the answer is NOT specific enough for the direction. We MUST find the VALUE of the angle.



To find the value of the angle we use a Trig function called TANGENT.

$$\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{55}{95} = 0.5789$$

$$\theta = \tan^{-1}(0.5789) = 30^\circ$$

So the COMPLETE final answer is :

$109.8 \text{ km} @ 30^\circ \text{ N of E}$

$109.8 \text{ km} < 30$

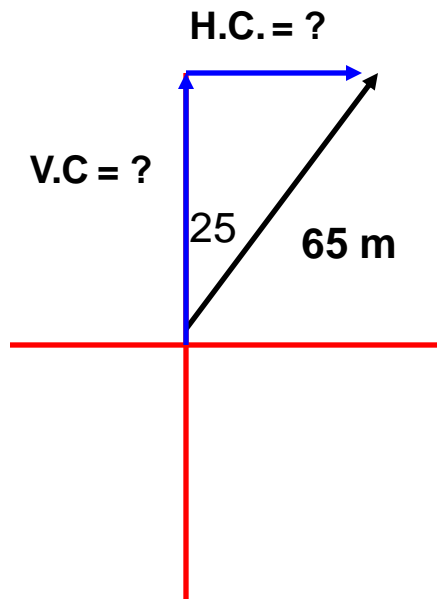
$95 \hat{i} \text{ km} + 55 \hat{j} \text{ km}$

What if you are missing a component?

Suppose a person walked 65 m, 25 degrees East of North. What were his horizontal and vertical components?

The goal: **ALWAYS MAKE A RIGHT TRIANGLE!**

To solve for components, we often use the trig functions sine and cosine.



$$\cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \quad \sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\text{adj} = \text{hyp} \cos \theta$$

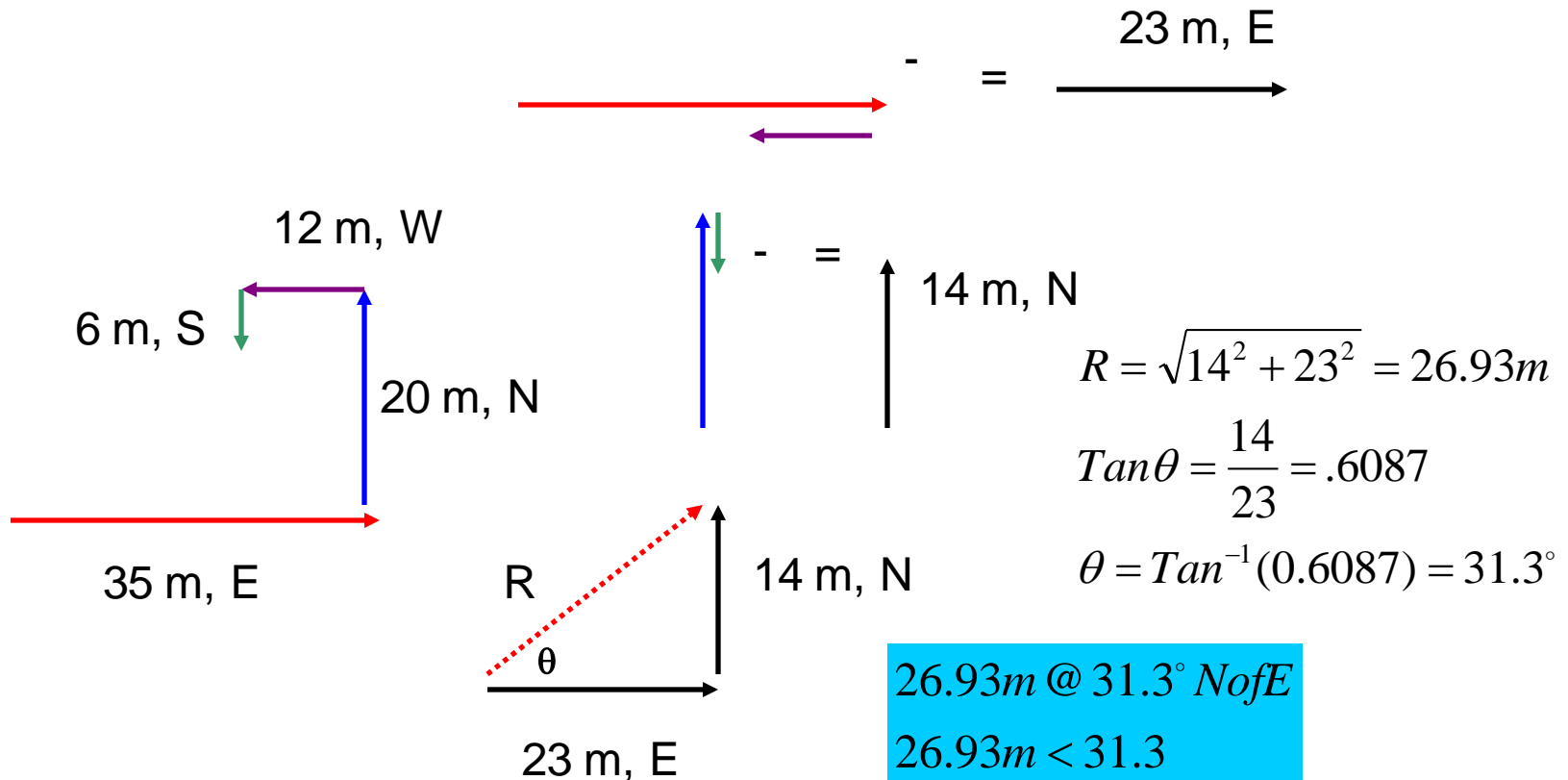
$$\text{opp} = \text{hyp} \sin \theta$$

$$\text{adj} = V.C. = 65 \cos 25 = 58.91 \text{ m, N or } 58.91 \hat{j} \text{ m}$$

$$\text{opp} = H.C. = 65 \sin 25 = 27.47 \text{ m, E or } 27.47 \hat{i} \text{ m}$$

Example

A bear, searching for food wanders 35 meters east then 20 meters north. Frustrated, he wanders another 12 meters west then 6 meters south. Calculate the bear's displacement.

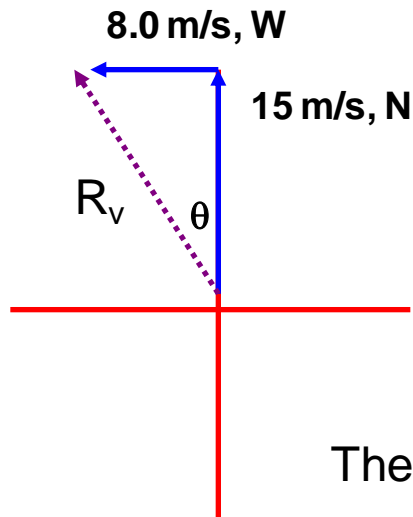


The Final Answer:

$26.93 \text{ m @ } 31.3^\circ \text{ N of E}$
 $26.93 \text{ m} < 31.3$
 $23\hat{i} \text{ m} + 14\hat{j} \text{ m}$

Example

A boat moves with a velocity of 15 m/s, N in a river which flows with a velocity of 8.0 m/s, west. Calculate the boat's resultant velocity with respect to due north.



The Final Answer :

$$R_v = \sqrt{8^2 + 15^2} = 17 \text{ m/s}$$

$$\tan \theta = \frac{8}{15} = 0.5333$$

$$\theta = \tan^{-1}(0.5333) = 28.1^\circ$$

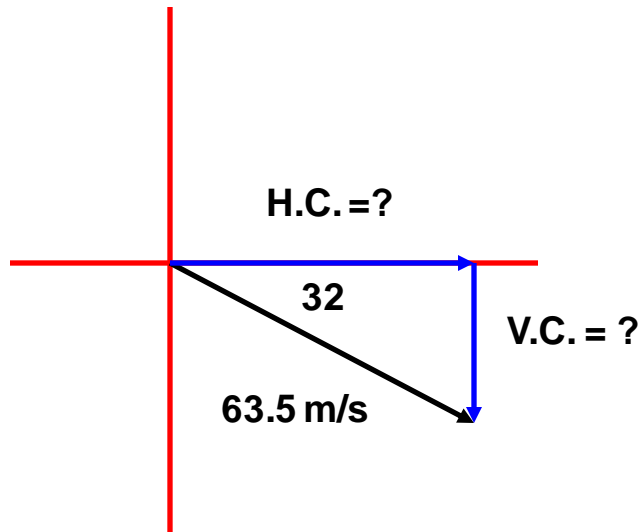
$17 \text{ m/s @ } 28.1^\circ \text{ W of N}$

$17 \text{ m/s} < 118.1^\circ$

$-8\hat{i} \text{ m/s} + 15\hat{j} \text{ m/s}$

Example

A plane moves with a velocity of 63.5 m/s at 32 degrees South of East. Calculate the plane's horizontal and vertical velocity components.



$$\cosine \theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \quad \text{sine } \theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$adj = hyp \cos \theta$$

$$opp = hyp \sin \theta$$

$$adj = H.C. = 63.5 \cos 32 = 53.85 \text{ m/s, } E \text{ or } 53.85 \hat{i}$$

$$opp = V.C. = 63.5 \sin 32 = 33.64 \text{ m/s, } S \text{ or } -33.64 \hat{j}$$

The “Dot” Product (Vector Multiplication)

Multiplying 2 vectors sometimes gives you a SCALAR quantity which we call the **SCALAR DOT PRODUCT**.

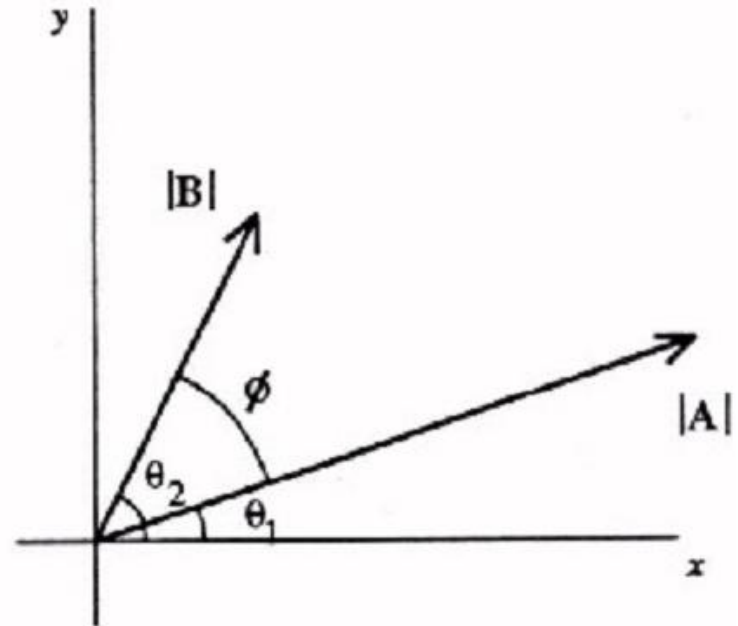
In polar notation consider 2 vectors:

$$\mathbf{A} = |\mathbf{A}| \angle \theta_1 \quad \& \quad \mathbf{B} = |\mathbf{B}| \angle \theta_2$$

The dot product between A and B produces a SCALAR quantity. The magnitude of the scalar product is defined as:

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \phi$$

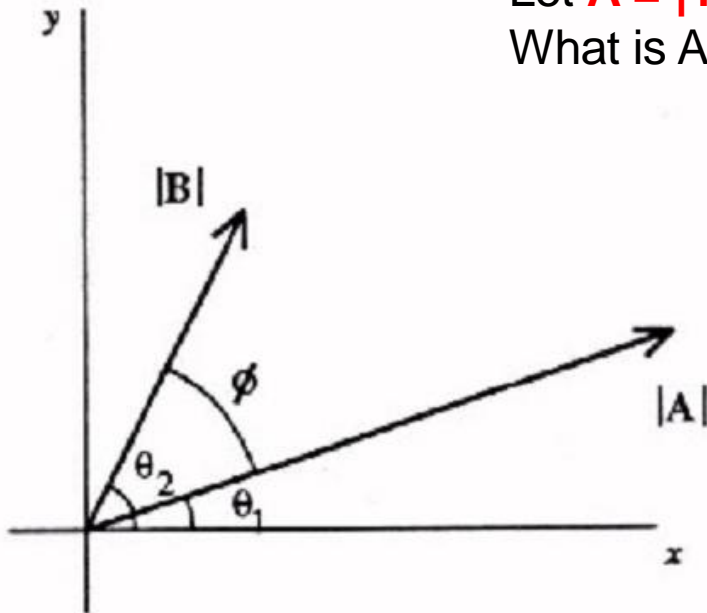
Where θ is the NET angle between the two vectors. As shown in the figure.



The Scalar Dot Product

Let $A = |12| < 30$, Let $B = |5| < 65$

What is A "dot" B?



$$A \bullet B = |A||B| \cos \theta = |12||5| \cos 35$$

$$A \bullet B = 49.15$$

In unit vector notation, it looks a little different. Consider:

$$\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$$

$$\mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}$$

The "Dot" product between these is equal to:

$$A \bullet B = A_x B_x + A_y B_y + A_z B_z$$

The Scalar Dot Product

Example:

$$\text{Let } \mathbf{A} = (3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k})$$

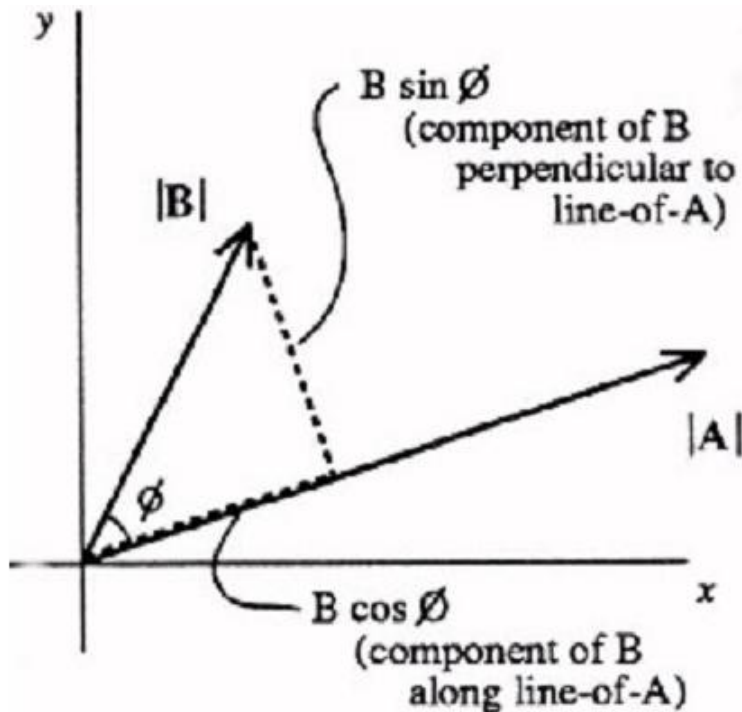
$$\text{Let } \mathbf{B} = (2\mathbf{i} + 7\mathbf{j} + 3\mathbf{k})$$

$$\text{Therefore, } \mathbf{A} \cdot \mathbf{B} = (3)(2) + (-4)(7) + (-5)(3) = -37$$

What is the **SIGNIFICANCE** of the dot product?

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \phi$$

The significance of the dot product



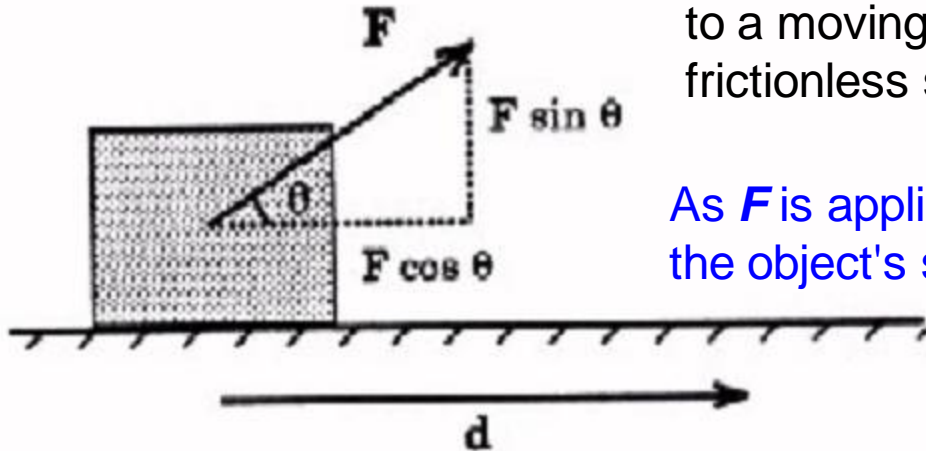
In this figure, vector B has been split into 2 components, one PARALLEL to vector A and one PERPENDICULAR to vector A.

Notice that the component parallel to vector A has a magnitude of $|B|\cos \theta$

THEREFORE when you find the DOT PRODUCT, the result is:

- i) The MAGNITUDE of one vector, in this case $|A|$ and,
- ii) The MAGNITUDE of the 2nd vector's component that runs **parallel** to the first vector. (That is where the cosine comes from)

Dot Products in Physics



Consider this situation: A force F is applied to a moving object as it transverses over a frictionless surface for a displacement, d .

As F is applied to the object it will increase the object's speed!

But which **part** of F really causes the object to increase in speed?

It is **$|F| \cos \theta$** ! Because it is **parallel** to the displacement d

In fact if you apply the dot product, you get **$(|F| \cos \theta)d$** , which happens to be defined as "**WORK**" (check your equation sheet!)

$$A \bullet B = |A||B| \cos \theta$$

$$W = F \bullet x = |F||x| \cos \theta$$

Work is a type of energy and energy **DOES NOT** have a direction, that is why WORK is a scalar or in this case a SCALAR PRODUCT (AKA DOT PRODUCT).

The “Cross” Product (Vector Multiplication)

Multiplying 2 vectors sometimes gives you a VECTOR quantity which we call the **VECTOR CROSS PRODUCT**.

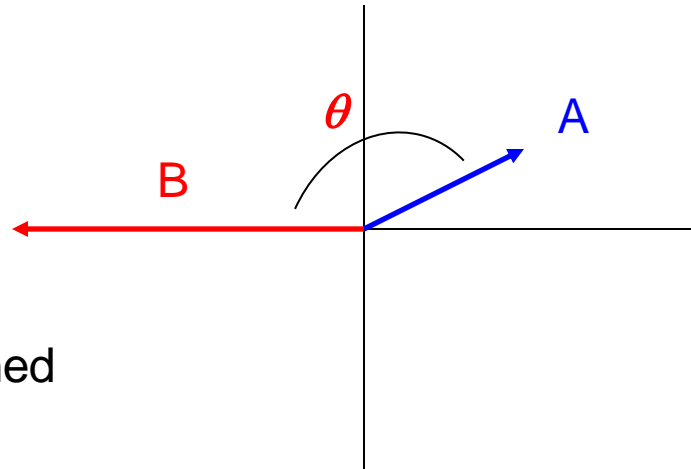
In polar notation consider 2 vectors:

$$\mathbf{A} = |\mathbf{A}| \angle \theta_1 \quad \& \quad \mathbf{B} = |\mathbf{B}| \angle \theta_2$$

The cross product between A and B produces a VECTOR quantity. The magnitude of the vector product is defined as:

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}| |\mathbf{B}| \sin \phi$$

Where θ is the NET angle between the two vectors. As shown in the figure.



The Vector Cross Product

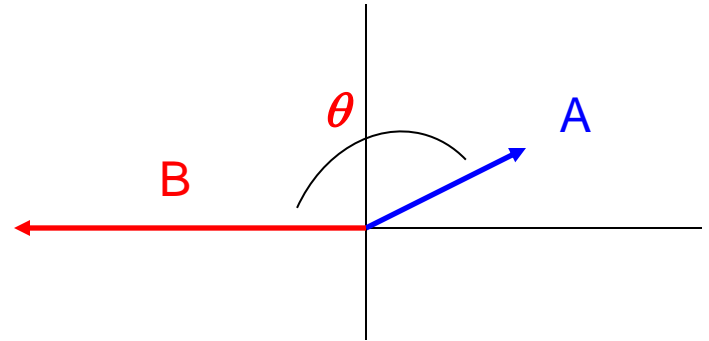
Example:

Let $A = |5| < 30$, Let $B = |12| < 180$

What is A "cross" B ?

$$A \otimes B = |A||B|\sin\theta = |12||5|\sin 150$$

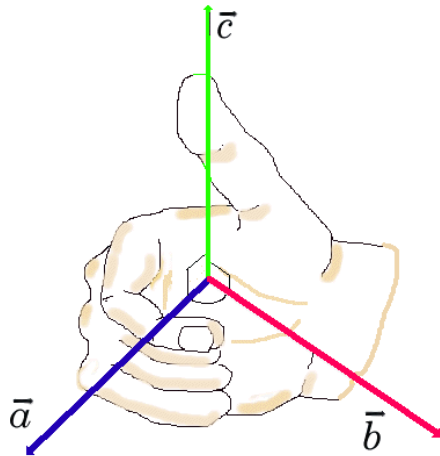
$$A \otimes B = 30\hat{k}$$



What about the direction???? Positive \hat{k} ???

We can use what is called the **RIGHT HAND THUMB RULE**.

- Fingers are the first vector, A
- Palm is the second vector, B
- Thumb is the direction of the cross product.**
- Cross your fingers, A , towards, B so that they CURL. The direction it moves will be either **clockwise (NEGATIVE)** or **counter clockwise (POSITIVE)**



In our example, the thumb points **OUTWARD** which is the Z axis and thus our answer would be $30 \hat{k}$ since the curl moves counter clockwise.

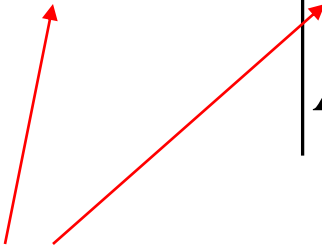
Cross Products and Unit Vectors

Consider:

$$\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$$

$$\mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}$$

The cross product between B and A produces a VECTOR of which a 3x3 matrix is need to evaluate the magnitude and direction.

$$\mathbf{B} \otimes \mathbf{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix}$$


Since **B** is the first vector it comes first in the matrix

You start by making a 3x3 matrix with 3 columns, one for i, j, & k-hat. The components then go under each appropriate column.

Cross Products and Unit Vectors

$$B \otimes A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix}$$

You then make an **X** in the columns OTHER THAN the unit vectors you are working with.

- For “i”, cross **j x k**
- For “j”, cross **i x k**
- For “k”, cross **i x j**

Let's start with the i-hat vector: **We cross j x k**

$$\hat{i} = (B_y A_z) - (B_z A_y)$$

Now the j-hat vector: **We cross i x k**

$$\hat{j} = (B_z A_x) - (B_x A_z)$$

Now the k-hat vector: **We cross i x j**

$$\hat{k} = (B_x A_y) - (B_y A_x)$$

Example

Example: Let $\mathbf{A} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ and $\mathbf{B} = -2\mathbf{i} - 4\mathbf{j} - 6\mathbf{k}$. What is $\mathbf{B} \times \mathbf{A}$?

$$\mathbf{B} \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -4 & -6 \\ 3 & -4 & 5 \end{vmatrix}$$

Let's start with the i-hat vector: We cross $\mathbf{j} \times \mathbf{k}$

$$\hat{i} = (-4)(5) - (-6)(-4) = -44$$

Now the j-hat vector: We cross $\mathbf{i} \times \mathbf{k}$

$$\hat{j} = (-6)(3) - (-2)(5) = -8$$

Now the k-hat vector: We cross $\mathbf{i} \times \mathbf{j}$

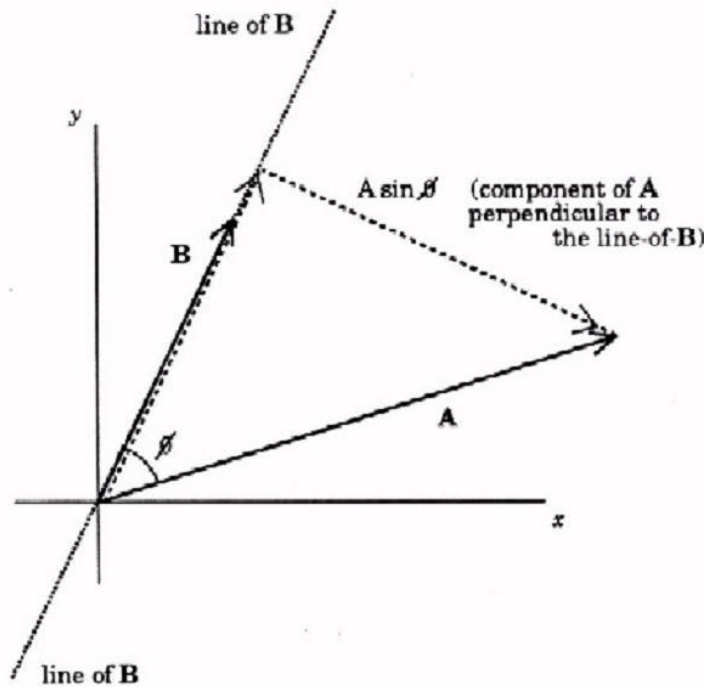
$$\hat{k} = (-2)(-4) - (-4)(3) = 20$$

The final answer would be:

$$\mathbf{B} \otimes \mathbf{A} = -44\hat{i} - 8\hat{j} + 20\hat{k}$$

The significance of the cross product

$$|A \times B| = |A||B|\sin \phi$$



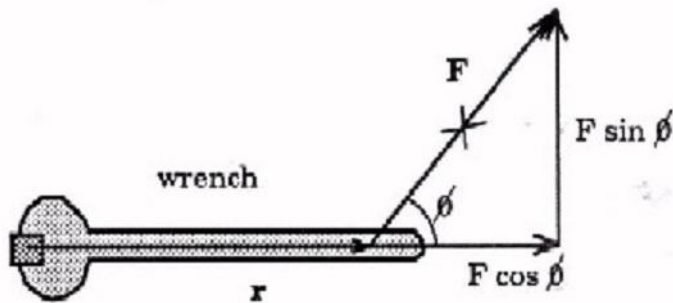
In this figure, vector A has been split into 2 components, one PARALLEL to vector B and one **PERPENDICULAR** to vector B. Notice that the component perpendicular to vector B has a magnitude of $|A|\sin \theta$

THEREFORE when you find the CROSS PRODUCT, the result is:

- i) The MAGNITUDE of one vector, in this case $|B|$ and,
- ii) The MAGNITUDE of the 2nd vector's component that runs **perpendicular** to the first vector. (that is where the sine comes from)

Cross Products in Physics

There are many cross products in physics. You will see the matrix system when you learn to analyze circuits with multiple batteries. The cross product system will also be used in mechanics (rotation) as well as understanding the behavior of particles in magnetic fields.



A force F is applied to a wrench a displacement r from a specific point of rotation (ie. a bolt).

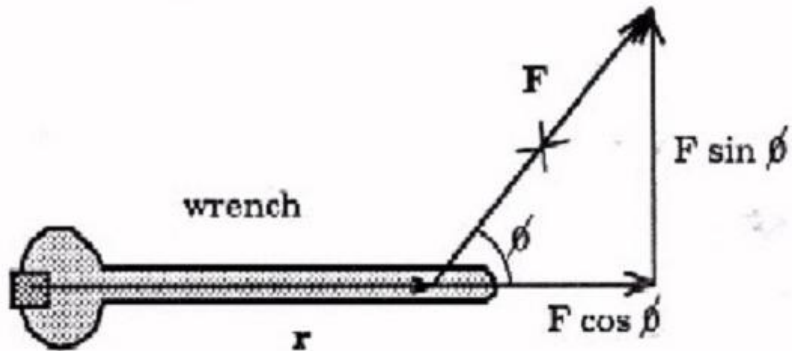
Common sense will tell us the larger r is the easier it will be to turn the bolt.

But which part of F actually causes the wrench to turn? $|F| \sin \theta$

$$A \otimes B = |A||B| \sin \theta$$

$$\vec{F} \otimes \vec{r} = |F||r| \sin \theta$$

Cross Products in Physics



$$A \otimes B = |A||B| \sin \theta$$

$$\vec{F} \otimes \vec{r} = |F||r| \sin \theta$$

What about the DIRECTION?

Which way will the wrench turn? **Counter Clockwise**

Is the turning direction positive or negative? **Positive**

Which way will the BOLT move? IN or OUT of the page? **OUT**

You have to remember that cross products give you a direction on the OTHER axis from the 2 you are crossing. So if “r” is on the x-axis and “F” is on the y-axis, the cross products direction is on the z-axis. **In this case, a POSITIVE k-hat.**